Weakly disordered absorbing-state phase transitions

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The effects of quenched disorder on nonequilibrium phase transitions in the directed percolation universality class are revisited. Using a strong-disorder energy-space renormalization-group method, it is shown that for any amount of disorder the critical behavior is controlled by an infinite-randomness fixed point in the same universality class of the random transverse-field Ising models.

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Using the formalism and the knowledge of equilibrium phase transitions, a natural trend with the aim of establishing and classifying possible universality classes in nonequilibrium transitions arose [1,2]. It was conjectured that the critical behavior of short-ranged interacting models with scalar order parameter and absence of conservation laws and extra symmetries are in the directed percolation universality class [3–5], which separates an active fluctuating state from an inactive (absorbing) nonfluctuating one [6]. Examples include transitions in the contact process [7], catalytic reactions [8], depinning interface growth [9,10], and marginal growth of turbulent domains in laminar flows [11].

Despite the theoretical understanding on the ubiquitous directed percolation universality class, its critical exponents have hardly been seen in real experiments [12] (see, however, Ref. [13]). It was then suspected that quenched disorder may be responsible. For spatial dimension $d<4$, this is indeed the case as dictated by the Harris criterion [14–16] and confirmed by field-theoretical methods [17], which showed that the renormalization-group equations have only runaway solutions towards large disorder. In addition, disorder-dependent Griffiths-like phases [18,19] nearby criticality have been observed [16,20–25].

This scenario thus points out an unconventional critical behavior originating from the interplay between *large* spatial disorder fluctuations and strong correlations. Motivated by this reasoning, a strong-disorder renormalization-group (SDRG) method [26,27] was applied to the random contact process model [28]. For *strong* disorder, the critical behavior is governed by a universal infinite-randomness fixed point (IRFP) in the same universality class of the random transverse-field Ising model, whose dynamical scaling is known to be activated [29–31], i.e., length ξ and time τ are related through $\ln \tau \sim \xi^{\psi}$, with ψ (dubbed tunneling exponent) being universal. For *weak* disorder, on the other hand, the critical point has finite disorder and usual power-law scaling $\tau \sim \xi^{z}$ with nonuniversal dynamical exponent z proportional to the disorder strength and is formally infinite at the transition between the weak- and strong-disorder limits. These conclusions were also supported by density-matrix renormalization-group calculations in $d=1$ [28]. Further Monte Carlo calculations in $d=2$ confirmed the above scenario. However, the possibility that the weak-disorder regime was an artifact of finite-size effects was raised [32].

Facing the logarithmically slow dynamics, large-scale Monte Carlo simulations in $d=1$ for system sizes up to 10^7 sites and times up to 10^9 were performed [33]. No nonuniversal weak-disorder critical regime was found, shrinking considerably the parameter space in which it would exist and, together with the field-theoretical results, strongly suggesting its nonexistence. It then raises the following puzzle. How can the SDRG suggest a *finite*-disordered fixed point while Monte Carlo simulations point to an *infinite*-disordered one? Since the SDRG method is devised to include any minimal effects of disorder, it should be able to capture the physics of any IRFP as well as to point out its existence.

This Brief Report is devoted to solve this question. In generalizing the SDRG method, we show that the critical system is governed by a universal IRFP when *any* amount of disorder is present. Moreover, our motivation goes beyond the issue of settling the correct universality class of weakly disordered absorbing-state phase transitions. It deals with the delicate issue of implementing a SDRG in such a limit, which is an important tool to tackle many disordered systems.

For definiteness, we now introduce the system, review the usual SDRG for random contact process [28], point out its failure, and modify it in order to overcome this problem.

The contact process can be defined in a lattice in which each site i can have either a healed ($\sigma_i=1$) or an infected ($\sigma_i=-1$) particle. A healed particle at site i can be contaminated by an infected one in a neighboring site j at rate $\lambda_{ij}=\lambda_{ji}$ [44]. Also, an infected particle at site i can get spontaneously healed at rate μ_i. The system has a stochastic dynamics governed by a master equation $\partial_t \mathbf{P}(\{\sigma\},t)=-H\mathbf{P}(\{\sigma\},t)$ where the vector $\mathbf{P}$ gives the probability of finding the configuration $\{\sigma\}=(\sigma_1,\sigma_2,\ldots)$ at time t and

$$H=\sum_i \mu_i M_i+\sum_{\langle i,j\rangle}\lambda_{ij}(n_iQ_j+Q_in_j) \tag{1}$$

is the generator of the Markov process [32,34,35]. Here,

$$M=\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix},\quad n=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad Q=\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix},$$

and $\langle i,j\rangle$ restricts the sum to nearest neighbors only.

The usefulness of this "quantum Hamiltonian formalism" comes from the fact that the steady state probability distribution $\mathbf{P}(\{\sigma\},t\to\infty)$ coincides with ground state of H and that the long-time relaxation properties are obtained from the low-lying spectrum of H. Although H is in general non-Hermitian, some standard methods can still be used.

For the disordered case, λ_{ij} and μ_i are random independent variables distributed according to $P(\lambda)$ and $R(\mu)$, respectively. In this case, the low-lying spectrum of H can be reached by the following recipe (for simplicity, we focus on the $d=1$ case): (i) search for the fastest ("high-energy") scale in the system $\Omega=\max\{\lambda_i,\mu_i\}$, (ii) integrate out locally the corresponding mode, and (iii) renormalize the remaining degrees of freedom. Those steps are the basis of the SDRG method [36].

When (ii.a) $\Omega=\lambda_2$, particles on sites 2 and 3 can be considered as one since they will be mostly in the same state, i.e., either both healed or both infected. Then, (iii.a) one treats $H_0=\lambda_2(n_2Q_3+Q_2n_3)$ exactly and $H_1=\mu_2M_2+\mu_3M_3$ as a perturbation. H_0 has two twofold multiplets. In the ground (excited) one, particles 2 and 3 are in the same (opposite) state. H_1 lifts the degeneracy of the ground multiplet, which corresponds to the effective healing rate $\tilde{\mu}$ of particle cluster 2 and 3. In second order of perturbation theory, one finds that $\tilde{H}_1=\tilde{\mu}\tilde{M}$, with

$$\tilde{\mu}=\kappa_\mu\mu_2\mu_3/\lambda_2,\quad \text{with } \kappa_\mu=2. \tag{2}$$

When (ii.b) $\Omega=\mu_2$, the particle at site 2 can be considered as healed for all times. Hence (iii.b) one treats $H_0=\mu_2M_2$ exactly and $H_1=\lambda_1(n_1Q_2+Q_1n_2)+\lambda_2(n_2Q_3+Q_2n_3)$ perturbatively. H_0 has two fourfold multiplets. The ground (excited) one refers to particle 2 healed (infected). H_1 then lifts the degeneracy which corresponds to an effective infection rate $\tilde{\lambda}$ between particles 1 and 3. In second order of perturbation theory $\tilde{H}_1=\tilde{\lambda}(n_1Q_3+Q_1n_3)$, with

$$\tilde{\lambda}=\kappa_\lambda\lambda_1\lambda_2/\mu_2,\quad \text{with } \kappa_\lambda=1. \tag{3}$$

Once set the recursion relations (2) and (3), flow equations for $P(\lambda)$ and $R(\mu)$ can be constructed and the fixed-point distributions obtained [30,32]. In principle, this gives the long-time behavior of the system. The multiplicative structure of Eqs. (2) and (3) is very important. Under these transformations, $P(\lambda)$ and $R(\mu)$ become indefinitely broad at criticality for any amount of disorder as long as $0<\kappa_{\mu,\lambda}\leqslant 1$ [30,32,37]. However, for $\kappa_{\mu,\lambda}>1$ the SDRG becomes inconsistent for weak disorder because the renormalized couplings are typically bigger than the decimated ones. It is thus tempting to interpret this result as a runaway flow towards weak disorder in odds with the field-theoretical results [17]. As we show below, this is not the case. The generation of a transition rate larger than the decimated ones is unphysical. The numerical prefactor $\kappa_\mu>1$ is just an artifact of treating H_1 until second order in perturbation theory.

According to Eq. (2), the splitting of the ground multiplet of H_0 due to H_1 may overcome the distance (λ_2) between the two unperturbed multiplets for certain values of $\mu_{2,3}$ even though $\mu_{2,3}<\lambda_2$. Treating H_0+H_1 exactly, however, this can never be the case. The ground state has energy 0 and the excited ones are solutions of the polynomial

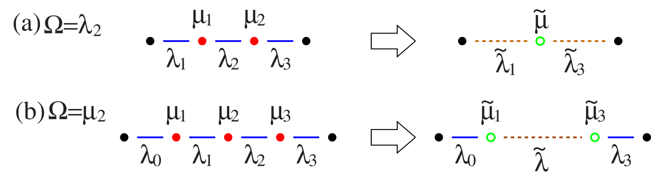

FIG. 1. (Color online) Schematic decimation procedure.

$$x^3-2\xi x^2+(\xi^2+\mu_2\mu_3)x-\mu_2\mu_3(\lambda_2+\xi)=0, \tag{4}$$

with $\xi=\lambda_2+\mu_2+\mu_3$. The renormalized healing rate $\tilde{\mu}$ is thus its minimal root. Although we could not solve $\tilde{\mu}$ analytically, its maximum value is shown to be $\tilde{\mu}_{\max}=(2-\sqrt{2})\lambda_2$, which happens for $\mu_3=\mu_2=\lambda_2$. Numerical inspections of Eq. (4) show that $\tilde{\mu}\leqslant\min\{\mu_2,\mu_3,\lambda_2\}$ in general.

In addition, the operators connecting this particle cluster to the rest of the chain have also to be projected onto the same states. We find that $n_{2,3}=\alpha_{2,3}\tilde{n}$ and $Q_{2,3}=\alpha_{2,3}\tilde{Q}$, where $(1+c_2+c_3)\alpha_{2,3}=1+c_{2,3}$, with $(\lambda_2+\mu_{2,3}-\tilde{\mu})^2c_{2,3}=\lambda_2\mu_{3,2}$. (Note that $1/2\leqslant\alpha_{2,3}\leqslant 1$.) Therefore the SDRG decimation procedure summarizes in replacing $\Sigma_{i=1,3}\lambda_i(n_iQ_{i+1}+Q_in_{i+1})+\Sigma_{i=2,3}\mu_iM_i$ by $\tilde{\lambda}_1(n_1\tilde{Q}+Q_1\tilde{n})+\tilde{\mu}\tilde{Q}+\tilde{\lambda}_3(\tilde{n}Q_4+\tilde{Q}n_4)$ with $\tilde{\lambda}_{1,3}=\alpha_{2,3}\lambda_{1,3}$ (see Fig. 1(a)). The renormalization of $\lambda_{1,3}$ is not considered in the usual perturbative SDRG which is indeed a "weaker" effect since $\alpha_{2,3}\in[1/2,1]$ and approaches 1 in the strong-disorder limit.

Repeating the same procedure when decimating a healing rate, Eq. (3) is then replaced by

$$\tilde{\lambda}=\zeta-\chi, \tag{5}$$

with $2\zeta=\lambda_1+\lambda_2+\mu_2$ and $\chi=\sqrt{\zeta^2-\lambda_1\lambda_2}$, implying $\tilde{\lambda}\leqslant\min\{\lambda_1,\lambda_2,\mu_2\}$. [Its maximal value $\tilde{\lambda}_{\max}=\mu_2(3-\sqrt{5})/2$ happens for $\lambda_1=\lambda_2=\mu_2$.] Moreover, $M_{1,3}=\beta_{1,3}\tilde{M}_{1,3}$, $n_{1,3}=\tilde{n}_{1,3}$ and $Q_{1,3}=\tilde{Q}_{1,3}$, with $4\beta_1\chi(\zeta+\chi)=\lambda_1(3\mu_2-\lambda_2)+(\lambda_2+\mu_2)(\mu_2+\lambda_2+2\chi)$ and β_3 is obtained by exchanging $\lambda_1\rightleftharpoons\lambda_2$ in β_1. (Note that $3/4\leqslant\beta_{1,3}\leqslant 1$.) These results mean we have to replace

$$\sum_{i=0,3}\lambda_i(n_iQ_{i+1}+Q_in_{i+1})+\sum_{i=1,3}\mu_iM_i$$

by

$$\lambda_0(n_0\tilde{Q}_1+Q_0\tilde{n}_1)+\tilde{\mu}_1\tilde{M}_1+\tilde{\lambda}(\tilde{n}_1\tilde{Q}_2+\tilde{Q}_1\tilde{n}_2)+\tilde{\mu}_3\tilde{M}_3 + \lambda_3(\tilde{n}_3Q_4+\tilde{Q}_3n_4),$$

where $\tilde{\mu}_{1,3}=\beta_{1,3}\mu_{1,3}$ (see Fig. 1(b)). Note that $n_{1,3}$ $(Q_{1,3})$ has no projection onto $\tilde{n}_{3,1}$ $(\tilde{Q}_{3,1})$. If this was not the case, the technical treatment of this SDRG would be more difficult because further-nearest-neighbor interactions would arise. Long-ranged interactions may point out delocalized states. Their absence suggests that the SDRG here presented is amenable.

Importantly, there are no level crossings in the entire region where the parameters of H_1 are less than or equal to the parameters of H_0, meaning the interpretation of the decimation steps still holds. Also important, the energy difference between the second and first excited multiplets (Δ_{21}) of H_0+H_1 only increases when increasing the perturbation and is always greater than the energy difference between the first excited and ground multiplets (Δ_{10}). Precisely, $\Delta_{10}\leqslant(1-1/\sqrt{2})\Delta_{21}$.

Therefore exactly projecting the entire system in the low-energy states of H_0+H_1 makes the renormalization-group approach totally consistent. Whether or not these new recursion relations drive the system to the universal IRFP is not straightforwardly clear. This is the question we address in the next part of this paper.

The fate of the critical point is obtained by solving the standard flow equations [30,32] for $P(\lambda)$ and $R(\mu)$ with the perturbed renormalized rates (2) and (3) replaced by their exact counterparts (4) and (5) in addition to the weaker renormalization of the neighboring transition rates ($\tilde{\lambda}_{1,3}$ and $\tilde{\mu}_{1,3}$ in Fig. 1). Because of the complicated analytical structure of these quantities, a detailed analytical solution is hampered. We then rewrite $\tilde{\mu}=\kappa'_{\mu}\mu_2\mu_3/\lambda_2$ and $\tilde{\lambda}=\kappa'_{\lambda}\lambda_1\lambda_2/\mu_2$, where $\kappa'_{\mu,\lambda}$ are functions of the decimated transition rates. Moreover, we will neglect the renormalizations of $\tilde{\lambda}_{1,3}$ and $\tilde{\mu}_{1,3}$ [45]. Now, recall that (i) $\tilde{\lambda}$ and $\tilde{\mu}$ are always less than the decimated ones and that (ii) there is no correction to H_0 in first order of perturbation theory. Point (i) permits us to set $\kappa'_{\mu,\lambda}=1$ in the weak-disorder limit. Hence the system rapidly flows towards stronger disorder. As intermediate disorder is reached, the only way of stopping its further growth is making *all* decimations of type $\tilde{\mu}=\text{const}\times\mu_2$ [38], which corresponds to corrections in first order of perturbation theory. Point (ii) thus guarantees there is no hindrance on the flow towards even stronger disorder, in which limit $\kappa'_{\mu,\lambda}$ can be neglected [37]. We thus finally conclude that any amount of disorder drives the critical system towards the universal infinite-randomness fixed point.

This conclusion was checked by numerical implementation of the SDRG for weak- and moderate-disordered chains. (For consistency with the above proof, the weaker corrections to $\tilde{\lambda}_{1,3}$ and $\tilde{\mu}_{1,3}$ were neglected [46].) Following time τ and length ξ scales along the critical SDRG flow, the predicted [29] tunneling exponent $\psi=1/2$ was confirmed (see Fig. 2). Here, $\tau^{-1}=\Omega$ and ξ^{-1} is the density of active particle clusters. The off-critical Griffiths phases surrounding the critical point in which $\tau\sim\xi^{z}$ with disorder-dependent dynamical exponent is also confirmed in our numerics [47].

We now address the issue of weak disorder in higher dimensions. One key feature hinders a straightforward generalization of the RG steps here proposed: the reconnection of the lattice. Because the coordination number increases in $d>1$, one eventually needs to treat exactly big particle clusters. Leaving this task open, we cannot guarantee that all the RG steps will consistently lower the energy scale and drive the system to an IRFP. However, and somewhat surprisingly, it was shown that the lattice reconnection does not hinder the flow towards infinite randomness [31]. As the inconsistency of the simple recursion relations is just an artifact of the perturbative treatment, it is then reasonable to conclude that the RG flows of the directed percolation and the transverse-field Ising universality classes are the same in the presence of

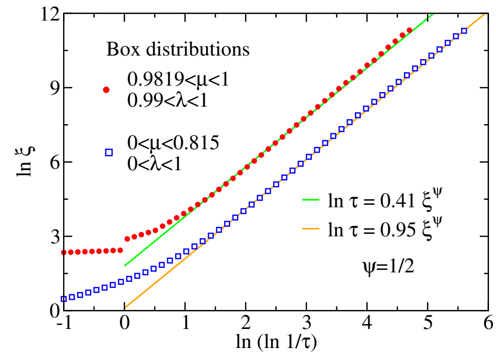

FIG. 2. (Color online) Time τ and length ξ scales along the SDRG flow at criticality. Transition rates are drawn from boxlike distributions as indicated. Chains have 2×10^6 sites and the data were averaged over 100 disorder realizations. Error bars are about the symbol size.

disorder in $d=2$ and 3, as suspected in Ref. [32]. This would be in agreement with Monte Carlo simulations in $d=2$ [39], with the Harris criterion [14–16] and with the field-theoretical runaway flow solutions [17].

Recently, the experimental realization of the clean directed percolation universality class have been claimed [13,40]. These experiments now raise another puzzle in the face of our results and many others [14–17,32,33,39]. We would like to point out two crossovers which may give an explanation. One is the time crossover which was stressed in Ref. [33] (see, e.g., Fig. 7 therein). Because of the logarithmically slow dynamics, the "true" steady state takes place only after a long period of relaxation. The other one is the clean-dirty crossover length. As in spin chains [41,42], there is a crossover length below which disorder is irrelevant. The cleaner the sample the longer the crossover length which reaches hundreds of sites even for spin chains with moderate disorder. The time crossover is analogous to the temperature crossover in spin chains. Only at very low temperatures are the low-energy states important. The length crossover is equally analogous. Statistically rare fluctuations (the so-called large rare regions) only exist on large samples. Naturally, these crossovers are related through the dynamics. In Ref. [40], the system size is of order of hundreds of degrees of freedom. It is thus reasonable that the exponents measured are nonuniversal between the clean and the infinite-randomness fixed point. (This also may apply to other experiments [12].) The crossover length of the samples in Ref. [13] seems much bigger.

In the face of the possibility of explaining many experiments, it is thus desirable to study the aforementioned crossover of the exponents, which should be accomplished without much effort by Monte Carlo calculations in $d=1$, for instance. From the experimental side, it is desirable to pinpoint precisely the source of quenched disorder and to estimate its strength. Border effects may also diminish the effective size of the sample. Finally, due to the slow relaxation processes, time measurements have to be carefully taken when locating the critical point. These studies should shed considerable light on this problem.

In conclusion, we have modified the usual strong-disorder renormalization-group method in order to *exactly* recast the low-energy spectrum of the local fast-mode Hamiltonian. This allowed the method amenable to attack the problem in the weak-disorder limit in which the perturbative treatment yielded to runaway flow towards weak disorder. Applications to quantum spin chains as well as comparison with similar generalizations will be presented elsewhere.

As discussed in Refs. [30,37], this renormalization-group method is not justified in the weak-disorder limit. We, however, leave open the possibility that, by exactly projecting the entire Hamiltonian onto the local low-energy spectrum, the method will correctly point out whether weak disorder is irrelevant.

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[1] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England, 1999).
[2] R. Dickman, in *Nonequilibrium Statistical Mechanics in One Dimension*, edited by V. Privman (Cambridge University Press, Cambridge, England, 1997), Chap. 3.
[3] P. Grassberger and A. de la Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).
[4] H. K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981).
[5] P. Grassberger, Z. Phys. B: Condens. Matter **47**, 365 (1982).
[6] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[7] T. E. Harris, Ann. Probab. **2**, 969 (1974).
[8] R. M. Ziff, E. Gulari, and Y. Barshad, Phys. Rev. Lett. **56**, 2553 (1986).
[9] L.-H. Tang and H. Leschhorn, Phys. Rev. A **45**, R8309 (1992).
[10] S. V. Buldyrev, A.-L. Barabási, F. Caserta, S. Havlin, H. E. Stanley, and T. Vicsek, Phys. Rev. A **45**, R8313 (1992).
[11] Y. Pomeau, Physica D **23**, 3 (1986).
[12] H. Hinrichsen, Braz. J. Phys. **30**, 69 (2000).
[13] K. A. Takeuchi, M. Kuroda, H. Chate, and M. Sano, Phys. Rev. Lett. **99**, 234503 (2007).
[14] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[15] W. Kinzel, Z. Phys. B: Condens. Matter **58**, 229 (1985).
[16] A. J. Noest, Phys. Rev. Lett. **57**, 90 (1986).
[17] H. K. Janssen, Phys. Rev. E **55**, 6253 (1997).
[18] R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).
[19] B. M. McCoy, Phys. Rev. Lett. **23**, 383 (1969).
[20] A. J. Noest, Phys. Rev. B **38**, 2715 (1988).
[21] A. G. Moreira and R. Dickman, Phys. Rev. E **54**, R3090 (1996).
[22] R. Dickman and A. G. Moreira, Phys. Rev. E **57**, 1263 (1998).
[23] M. Bramson, R. Durrett, and R. H. Schonmann, Ann. Probab. **19**, 960 (1991).
[24] I. Webman *et al.*, Philos. Mag. B **77**, 1401 (1998).
[25] R. Cafiero, A. Gabrielli, and M. A. Muñoz, Phys. Rev. E **57**, 5060 (1998).
[26] S.-k. Ma, C. Dasgupta, and C.-k. Hu, Phys. Rev. Lett. **43**, 1434 (1979).
[27] C. Dasgupta and S.-k. Ma, Phys. Rev. B **22**, 1305 (1980).
[28] J. Hooyberghs, F. Iglói, and C. Vanderzande, Phys. Rev. Lett. **90**, 100601 (2003).
[29] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992).
[30] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).
[31] O. Motrunich, S. C. Mau, D. A. Huse, and D. S. Fisher, Phys. Rev. B **61**, 1160 (2000).
[32] J. Hooyberghs, F. Iglói, and C. Vanderzande, Phys. Rev. E **69**, 066140 (2004).
[33] T. Vojta and M. Dickison, Phys. Rev. E **72**, 036126 (2005).
[34] F. C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg, Ann. Phys. **230**, 250 (1994).
[35] G. Schütz, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 2001), Vol. 19, Chap. 1, pp. 1–251.
[36] F. Iglói and C. Monthus, Phys. Rep. **412**, 277 (2005).
[37] D. S. Fisher, Phys. Rev. B **50**, 3799 (1994).
[38] J. A. Hoyos and E. Miranda, Phys. Rev. B **70**, 180401(R) (2004).
[39] T. Vojta, A. Farquhar, and J. Mast (unpublished).
[40] P. Rupp, R. Richter, and I. Rehberg, Phys. Rev. E **67**, 036209 (2003).
[41] N. Laflorencie and H. Rieger, Phys. Rev. Lett. **91**, 229701 (2003).
[42] N. Laflorencie, H. Rieger, A. W. Sandvik, and P. Henelius, Phys. Rev. B **70**, 054430 (2004).
[43] T. Vojta, J. Phys. A **39**, R143 (2006).
[44] Note that we are not dividing λ_{ij} by the coordination number as usual.
[45] If the system flows towards strong disorder (as is the case), considering the renormalization of $\tilde{\lambda}_{1,3}$ and $\tilde{\mu}_{1,3}$ will only support this result.
[46] Their only effect is to shift the value of the critical point closer to its true value in addition to enhance the crossover length for weaker disorder. We remember that those nonuniversal quantities are not suitably computed by RG methods.
[47] For a review on the scaling of many observables, see Refs. [32,33,43].